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RESEARCH PAPER

Mathematical modeling of two-dimensional unconfined flow in aquifers

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ABSTRACT

Derivation of general equation for two-dimensional aquifer flow is given. In this derivation we perform a volume balance instead of a mass balance and obtained analytical solutions of two-dimensional saturated flow under various condition. We also constructed transient unconfined groundwater flow equation by combining continuity equation with the Darcy law and provide an analytical solution.

Key Words : Aquifer, Analytical solution, Unconfined, Two-dimensional, Transmissivity, Isotropic

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Flow in aquifers is often modeled as two-dimensional in the horizontal plane. This can be done because most aquifers have an aspect ratio like a thin pancake, with horizontal dimensions that are hundreds or thousands of times greater than their vertical thickness. In most aquifers, the bulk of the resistance encountered along a typical flow path is resistance to horizontal flow. When this is the case, the real three-dimensional flow system can be modeled in a reasonable way using a two-dimensional analysis. This is accomplished by assuming that h varies with x and y, but not with z, reducing the spatial dimensions of the mathematical problem to a horizontal plane. This simplifying assumption for modeling aquifer flow as horizontal two-dimensional flow is called the Dupuit-Forchheimer approximation, named after the French and German hydrologists who proposed and embellished the theory (Dupuit, 1863 and Forchheimer, 1886).

Dupuit and Forchheimer proposed the approximation for flow in unconfined aquifers, but the concept is equally applicable to confined aquifers with small amounts of vertical flow. They understood their approximation to mean that vertical flow was ignored. Kirkham (1967) later clarified the concept, pointing out that there may be vertical flow in Dupuit-Forchheimer models, but that resistance to vertical flow is neglected.

Dupuit-Forchheimer model represents in a physical sense, imagine an aquifer perforated by numerous tiny vertical lines that possess infinite hydraulic conductivity. The vertical lines eliminate the resistance to vertical flow,

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but the resistance to horizontal flow remains the same. In models using this approximation, the head distribution on any vertical line is hydrostatic $\left(\frac{\partial \mathbf{h}}{\partial \mathbf{z}} = \mathbf{0}\right)$.

Derivation:

First, equations will be derived for one-dimensional aquifer flow in the x direction and then they will be extended to two-dimensional flow in x, y plane. In this derivation we perform a volume balance instead of a mass balance.

Consider an elementary volume that is a vertical prism of cross-section Δx x Δy, extending the full saturated thickness of the aquifer b. First consider the discharge (volume/time) flowing through the face that is normal to the x axis as the left side of the prism. Using Darcy's law the flow (volume/time) into the prism at coordinate is Bath (1968).

$$-\mathbf{K_x}\,(\mathbf{x})\,\mathbf{b}\,(\mathbf{x})\,\Delta\mathbf{y}\frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}) \quad ....(1)$$

where, K_x (x) the hydraulic conductivity at coordinate x, b (x) is the saturated thickness at and x, and ∂h/∂x (x) is the x-direction component of the hydraulic gradient at x. For a uniform, single-layer aquifer, transmissivity is defined as T=Kb, so the above expression can be simplified to Boulton (1965).

$$-\mathbf{T_x}\,(\mathbf{x})\,\Delta\mathbf{y}\,\frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}) \quad ...(2)$$

where, T_x (x) is the x-direction transmissivity. Eq. (2) applies regardless of whether the aquifer consists of a single layer as in Eq. (1) or has some more complicated distribution of transmissivity such as multiple layers with varying K_x. The flow out of the right side of prism at coordinate **x +÷ax** is similarly defined as:

$$-\mathbf{T_x}\,(\mathbf{x}+\Delta\mathbf{x})\,\Delta\mathbf{y}\,\frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}+\Delta\mathbf{x}) \quad ....(3)$$

The net volume flux (volume/time) into the element through the top and bottom of the prism is given as:

$$\mathbf{N\Delta\,x\,\Delta y} \quad .....(4)$$

where, N is the net specific discharge coming in the top and bottom. The dimensions of N are volume/time/area [L/T]. The time rate of change in the volume of water stored in the element (volume/time) is:

$$\mathbf{S}\frac{\partial \mathbf{h}}{\partial \mathbf{t}}\mathbf{\Delta\,x\,\Delta y} \quad .....(5)$$

Balancing the volume fluxes given by the previous four expressions results in:

$$-\left[\mathbf{Tx\,(x)\,\Delta y\,\frac{\partial h}{\partial x}(x)}\right]+\left[\mathbf{Tx\,(x+\Delta x)\,\Delta y\,\frac{\partial h}{\partial x}(x+\Delta x)}\right]+\mathbf{N\Delta\,x\,\Delta y} = \mathbf{s}\frac{\partial \mathbf{h}}{\partial \mathbf{t}}\mathbf{\Delta\,x\,\Delta y} \quad ..(6)$$

Dividing by Δx Δy and then writing the limit for Δx approaching zero gives:

$$\underset{\mathbf{\Delta x \to 0}}{Iim}\left[\frac{\mathbf{Tx(x+\Delta x)\frac{\partial h}{\partial x}(x+\Delta x) - Tx(x)\frac{\partial h}{\partial x}(x)}}{\mathbf{\Delta x}}\right]+\mathbf{N} = \mathbf{S}\frac{\partial \mathbf{h}}{\partial \mathbf{t}}$$

The first term is a derivative. Therefore, this equation can be written more compactly as Bath (1968).

$$\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{Tx}\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right)+\mathbf{N} = \mathbf{S}\frac{\partial \mathbf{h}}{\partial \mathbf{t}} \quad ....(7)$$

This is the general equation for one-dimensional aquifer flow. It is founded on Darcy's law (Eq. 2 and 3) and conservation of mass Eq. (6).

Modeling of two-dimensional aquifer flow:

If we extend the derivation to two dimensional flows, the result is the general equation for two-dimensional aquifer flow, allowing for anisotropy and spatial variations in T Boulton (1965).

$$\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{Tx}\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right)+\frac{\partial}{\partial \mathbf{y}}\left(\mathbf{Ty}\frac{\partial \mathbf{h}}{\partial \mathbf{y}}\right)+\mathbf{N} = \mathbf{S}\frac{\partial \mathbf{h}}{\partial \mathbf{t}} \quad ....(8)$$

where, Tx,Ty are x and y direction transmissivities, N is net recharge or leakage, S is storage co-efficient, t is time.

Boundary and initial conditions:

In order to obtain a unique solution of partial differential equation corresponding to a given physical process, additional information about physical state of the process is required. This information is described by the boundary and initial conditions. For steady-state problem only boundary conditions are required, whereas for unsteady-state problem both boundary and initial conditions are required. Mathematically, the boundary condition includes geometry of the boundary and the values of the dependent variable or its derivative normal to the boundary. In physical terms, for ground-water applications the boundary conditions are generally of three types: (1) specified values, (2) specified flux or (3) value-dependent flux, where the value is head, concentration or temperature depending on the equation. The initial conditions are simply the values of the dependent variable specified every inside the boundary.

For example, in a confined aquifer for which the equations are linear, there is no need to impose the natural flow system since. In this case, the initial condition is drawdown (the dependent variable) equal to zero everywhere (Nguyen and Raudkivi,1983).

Transmissivity is isotropic and homogeneous:

If the transmissivity is isotropic and homogeneous ($T_x = T_y = T$ = constant), the Eq. (8) reduces to (Brutsaert *et al.*, 1971).

$$T\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right]+N=S\frac{\partial h}{\partial t} \quad ......(9)$$

This equation can be written more compactly by dividing by T and using the symbol for the Laplacian operator, we get:

$$\nabla^2 h+\frac{N}{T}=\frac{S}{T}\frac{\partial h}{\partial t} \quad ....(10)$$

If there is zero net recharge or leakage (N=0), then this becomes:

$$\nabla^2 h=\frac{S}{T}\frac{\partial h}{\partial t} \quad ....(11)$$

Steady-state flow with transmissivity is isotropic and homogeneous:

If flow is steady-state ($\frac{\partial h}{\partial t}=0$), the Eq. (10) takes the form Charles (2002).

$$\nabla^2 h=-\frac{N}{T} \quad ..... \quad (12)$$

Eq. (12) is known in physics and engineering as the Poisson equation, named after the French mathematician Denis Poisson (1781-1840). If flow is steady and there is zero net recharge/leakage (N =0), Eq. (12) reduces to the Laplace equation.

$$\nabla^2 h=0 \quad ....(13)$$

Flow in an unconfined aquifer:

Flow in an isotropic, homogeneous, unconfined aquifer with a horizontal impermeable base is a special case of aquifer flow. If we measure hydraulic head from the base of the aquifer, then h=b and T=Kh. Using this definition of transmissivity in Eq. (8) results in Boulton (1965).

$$K\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right]+N=S\frac{\partial h}{\partial t} \quad ....(14)$$

where, K is assumed to be isotropic and homogeneous. This is a nonlinear partial differential equation because the terms in parentheses involve h multiplied by its derivative, nonlinear equations are much more difficult to solve than linear ones. The nonlinear equation can be avoided if it is written in terms of the variable h^2 instead of h. This is done by substituting the following two relations (Bath, 1968),

$$h\frac{\partial h}{\partial x}=\frac{1}{2}\frac{\partial}{\partial x}\left(h^2\right) \quad ....(15)$$

and

$$h\frac{\partial h}{\partial y}=\frac{1}{2}\frac{\partial}{\partial y}\left(h^2\right) \quad ....(16)$$

Into Eq. (14), resulting in a differential equation in terms of h^2 :

$$\frac{K}{2}\left[\frac{\partial^2}{\partial x^2}\left(h^2\right)+\frac{\partial^2}{\partial y^2}\left(h^2\right)\right]+N=S\frac{\partial h}{\partial t} \quad ....(17)$$

Dividing by K/2, this reduces to:

$$\Delta^2(h)^2+\frac{2N}{K}=\frac{2S}{K}\frac{\partial h}{\partial t} \quad ....(18)$$

For steady flow this becomes the linear Poisson equation,

$$\Delta^2(h)^2=-\frac{2N}{K} \quad ....(19)$$

Flow is steady and N = 0 :

If flow is steady and there is zero net infiltration/ leakage (N=0) the general Eq. (19) reduces to the linear Laplace Eq. (De Wiest, 1969) :

$$\Delta^2(h)^2=0 \quad ....(20)$$

where, h must be measured from the horizontal aquifer base for Eq. 14 -20 to be valid.

Aquifers with uniform transmissivity:

For many flow problems in confined aquifers and some in unconfined aquifers, it is reasonable to construct a model that approximates the real system in the following ways: 1. The flow is steady state. 2. The resistance to vertical flow is neglected; only the resistance to horizontal flow is accounted for. 3. The aquifer transmissivity T is homogeneous and constant (Bath, 1968).

Solution for uniform flow:

One solution of Laplace Eq. (13) represents uniform flow in one direction, where the hydraulic gradient is constant over the whole x,y plane and the potentiometric surface is planar. On a large scale, the potentiometric surface of an aquifer is usually not planar. But if the area of interest is just a small portion of an aquifer, the head distribution within that area may be nearly planar and this solution can be useful. This solution can be derived by observing that one possible set of solutions for the Laplace equation would both (Yates, 1992).

$$\frac{\partial^2 \mathbf{h}}{\partial \mathbf{x}^2}=\mathbf{0} \text{ and } \frac{\partial^2 \mathbf{h}}{\partial \mathbf{y}^2}=\mathbf{0} \qquad .....(21)$$

If the above equations are true, then integration of the above gives:

$$\frac{\partial \mathbf{h}}{\partial \mathbf{x}}=\mathbf{A} \text{ and } \frac{\partial \mathbf{h}}{\partial \mathbf{y}}=\mathbf{B} \qquad ....(22)$$

where, A and B are constants. Integrating both of these equations results in a solution of the form (Bath, 1968).

$$\mathbf{h}=\mathbf{Ax}+\mathbf{By}+\mathbf{C} \qquad .....(23)$$

where, A,B and C are constants. This solution represents uniform horizontal flow with a planar potentiometric surface. The constants A and B are the hydraulic gradients in the x and y directions, as Eq. (22) shows. The constant C moves the head surface up and down to different elevations without affecting the gradient. By itself, this solution represents flow in a uniform direction with a uniform hydraulic gradient everywhere in the x, y plane. If A=B=0, this solution reduces to h = C, a stagnant condition with no gradient and no flow. Three points of known head are required to uniquely define the surface with the constants A,B and C.

Solution for radical flow to a well:

A very useful solution to Laplace's equation is that for steady radial flow, which applies to flow in the vicinity of a pumping well. This solution assumes radial flow toward a well, so it makes sense to formulate the solution in terms of a radial co-ordinate r centered on the well. The origin of the co-ordinate system is taken as the centerline of the well. With this solution, all flow is radially symmetric in the r direction. This solution for radial flow can be derived directly from the governing Laplace Eq. (13), or it can be derived by combining Darcy's law and mass balance. We will take the latter approach, which is straight forward Yates (1992).

Define the discharge of the well as $Q[L^3/T]$, which by convention here is positive for a well that removes water from the aquifer and negative for a well that inject water into the aquifer. With mass balance, this same discharge must be flowing through any closed boundary that can be drawn around the well. Imagine that this boundary is a cylinder of radius r centered on the well. The height of the aquifer is b, so the surface area that flow goes through on this cylinder is $2\pi rb$. The specific discharge in the negative r direction (towards the well) anywhere on this cylindrical surface is $-\mathbf{q}_r=\mathbf{K}\frac{\mathbf{dh}}{\mathbf{dr}}$. The total discharge through the cylinder is the product of specific discharge and the surface area of the cylinder, and it must equal the discharge of the well (Boulton,1965).

$$\mathbf{Q=2\text{®}rbK}\frac{\mathbf{dh}}{\mathbf{dr}}=\mathbf{2\text{®}rT}\frac{\mathbf{dh}}{\mathbf{dr}} \qquad ......(24)$$

This equation can be rearranged to separate the variables r and h to give:

$$\mathbf{dh}=\frac{\mathbf{Q}}{\mathbf{2\text{®}T}}\frac{\mathbf{dr}}{\mathbf{r}} \qquad .....(25)$$

Integrating both sides of this equation yields the solution for steady radial flow in an aquifer with constant T (Raudkivi, 1979).

$$\mathbf{h}=\frac{\mathbf{Q}}{\mathbf{2\text{®}\ T}}\mathbf{In\ r}+\mathbf{C} \qquad .....(26)$$

where, C is a constant and r is the radial distance from the center of the well to the point where h is evaluated. This solution satisfies Laplace's equation, which when written in terms of radial co-ordinates for radially symmetric flow is:

$$\boldsymbol{\Delta}^2\mathbf{h}=\frac{\partial^2\mathbf{h}}{\partial\mathbf{r}^2}+\frac{\mathbf{1}}{\mathbf{r}}\frac{\partial\mathbf{h}}{\partial\mathbf{r}} \qquad ......(27)$$

Because of the natural log in Eq. (26), the head predicted by this solution has the following behaviors close to and far from the well (James and Charles,1980).

As r ì =0|=h ì -ó and as r ì =+ó, h ì +ó ....(28)

Since wells always have some finite radius, the singular behaviour as r ì 0 is not a concern. On the other hand, the behaviour of this solution becomes inappropriate at large distances from the well. In real aquifers, heads do not increase indefinitely with distance from pumping wells because of the existence of features like rivers or lakes that supply water to the aquifer. Since this solution does not incorporate the influence of such far-field boundary conditions, its predictions become inaccurate far from the well. This solution alone is valid only in the region close to the well where the heads and discharges are dominated by the influence of the well. When the head is known at some point close to the well, the constant C in Eq. (26) can be determined. Say that the head at radius r_0 equals to h_0. The solution at $r = r_0$ is (Tsai and Chen,1996).

$$h_0 = \frac{Q}{2\pi T}\mathrm{In}\, r_0 + C \qquad ....(29)$$

Solving the above equation for C yields:

$$C = h_0 - \frac{Q}{2\pi T}\mathrm{In}\, r_0 \qquad .....(30)$$

Substituting this definition of C back into Eq. (26) gives a form of the solution for the case where head is known at a point near the well.

$$h = \frac{Q}{2\pi T}\mathrm{In}\frac{r}{r_0} + h_0 \qquad .....(31)$$

This equation is sometimes referred to as the Thiem equation (Thiem, 1906). The point where $r = r_0$ and $h = h_0$ can be at the radius of the pumping well if you know the head at the pumping well, or it can be at the location of some nearby non-pumping well or piezometer.

Solution for uniform recharge/leakage:

If there is steady flow and a nonzero net vertical flow in through the upper and lower boundaries of the aquifer ($N \neq 0$), Poisson's Eq. (12) applies. Then the recharge/leakage rate N is constant and independent on x, y, there are some fairly simple solutions that can be useful. One case where such a solution is often helpful is the recharge area of an unconfined aquifer. Another is a small portion of a confined aquifer where the net leakage through aquitards is approximately uniform (Singh, 2013).

The following function is a solution to the Poisson equation that models constant recharge/leakage at rate N over the entire x,y plane, as we will prove by differentiation (Bath, 1968):

$$h = -\frac{N}{2T}[Dx^2 + (1-D)y^2] + C \qquad ....(32)$$

where, D is positive constant in the range $0 \leq D \leq 1$. Performing the double differentiations on this solution proves that it is a solution of Poisson's equation for constant recharge/leakage at rate N (compare with Eq. (12).

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = -\frac{N}{2T}2D - \frac{N}{2T}2(1-D) = -\frac{N}{T} \qquad ....(33)$$

The head pattern produced by this solution for three different values of the constant D. If D =1/2, the recharge/leakage is conducted off to infinity in a radial flow pattern, and the contours of constant head are circles centered on the origin (Crank, 1975).

$$h = -\frac{N}{2T}\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) + C$$

$$h = -\frac{N}{4T}r^2 + C, \qquad (r^2 = x^2 + y^2) \qquad ...(34)$$

The hydraulic gradient increases with distance from the origin, which is necessary to conduct away an amount of recharge/leakage that increases with the square of distance from the origin. If D = 0 or D = 1, the flow pattern becomes one dimensional, with water flowing off to infinity in either the y or x direction, respectively. For other values of D, the head contours form ellipses, each with aspect ratio:

$$\frac{\Delta y}{\Delta x} = \sqrt{D/(1-D)} \qquad ...(35)$$

where, $\Delta y/\Delta x$ is the ratio of the y and x lengths of the ellipses. When $D < 1/2$, $\Delta y < \Delta x$ and when $D > 1/2$, $\Delta y < \Delta x$. Inverting Eq. (35) gives.

$$D = \frac{(\Delta y/\Delta x)^2}{1 + (\Delta y/\Delta x)^2} \qquad ...(36)$$

There is an infinite variety of solutions depending on the value of D because there is an infinite variety of possible lateral boundary condition for the case of uniform recharge/leakage. To see how various factors influence these solutions, examine the radially symmetric form, Eq. (34), for a constant T aquifer in a circular island in a lake. Assume the island has a radius r_0 and the head at the shore is h_0 Applying Eq. (34) at the shoreline yields.

$$h_0 = -\frac{N}{4T}r_0^2 + C \qquad ....(37)$$

Solving for C in the above gives:

$$C = h_0 + \frac{N}{4T}r_0^2 \qquad ....(38)$$

Substituting Eq. (38) back into Eq (34) gives the solution for this particular situation.

$$h = -\frac{N}{4T}(r^2 - r_0^2) + h_0 \qquad ....(39)$$

The head surface is a parabolic, radially symmetric mound with its highest level at the center of the island. The head surface is horizontal at the center of the island and gets steeper with increasing r. The height of the head above the lake level $(h - h_0)$ at the center of the island $(r = r_0)$ is.

$$h - h_0 = -\frac{N}{4T}r_0^2 \ (\text{at } r = 0) \qquad .....(40)$$

The height of the potentiometric surface is proportional to the ratio of recharge/leakage to transmissivity, N/T. when this ratio is higher; the mound in the potentiometric surface is higher. The height of the potentiometric surface is also proportional to r_0^2. The height of the mound is proportional to the square of the average distance to fixed-head boundaries (r_0^2 in the case). These concepts apply to aquifers of variable shape, not just to circular ones.

Solution of transient unconfined ground water flow:

The problem of unsteady flow of groundwater into a well has been extensively studied, but the equivalent two-dimensional problem of flow into a large excavation has not received the same attention. We give an analytical solution of the equations describing the transient unconfined groundwater flow into a large cut such as an open cut strip mine. Here the free surface boundary of flow is time dependent and is not known beforehand (Gambolati, 1976). The discharge comes from the elastic storage and also from the lowering of the water table. Besides the simplifying assumptions of negligible interrelation between the stress field and flow field of the aquifer and negligible effect of the capillary fringe (Brutsaert et *al.*, 1971) the treatments of unsteady confined flow usually assume that the elastic storage coefficient is negligible, *i.e.* the aquifer and water are incompressible, e.g. Gambolati (1976) and on an assumption related to the effect of falling water table on the vertical mass transport and the associated piezometric gradient. The Dupuit approximation neglects the vertical mass transport and puts the horizontal velocity proportional to the slope of the free surface (Boulton, 1954).

Governing equations in two dimensions:

The continuity equation combined with the Darcy law leads for two- dimensional unconfined flow to Eq. (11) *i.e.*

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} \qquad ....(41)$$

where, $h = \left(\frac{p}{\gamma}\right) + y$ is the piezometric head; T= Kh_0 is the transmissivity (m^2s^{-1}); h_0 is the undisturbed piezometric level and S is the storage co-efficient (Carslaw and Jaeger, 1959). For an unconfined aquifer the storage co-efficient S is given by Boulton (1965).

$$S = S_v + S_e \qquad ....(42)$$

where, S_v is the storage co-efficient due to partial drainage of voids with an upper limit of porosity of the aquifer; $S_e = h_0\, \rho_g\, (\alpha + n\beta)$ is the elastic storage co-efficient; α is the inverse of the modulus of elasticity E_s of the aquifer; β is the inverse of the bulk modulus E_w of water and n is the porosity. Since S_v is of the same order of size as n, the ratio of S_e to S_v is usually small, *i.e.*

$$S_e/n = h_0\, \rho[\beta + (\sim/n)] \qquad ....(43)$$

where, α and β are very small ($\beta \sim 5 \times 10^{-10}$ and α smaller still) and unless h_0 is extremely large the right hand side is of the order of 10^{-6}. For the saturated condition of the aquifer Eq. (41) reduces to (Necati ozisic , 1993).

$$\nabla^2 h = \frac{S_e}{T}\frac{\partial h}{\partial t} \qquad .... (44)$$

and for the incompressible aquifer and water to:

$$\nabla^2 h = 0 \qquad .....(45)$$

Eq. (45) is Laplace equation and which is valid for steady conditions and for the above unsteady conditions (Jacob Bear, 1979). The time dependence for this type of unsteady flow comes in through the upper boundary condition. The boundary conditions, are as follows:

(i) No flow across the lower impervious boundary:

$$\partial h / \partial y = 0 \text{ at } y = 0$$

(ii) At the free surface:

$$(Q_y - Q_x)\Delta t = S_v \Delta h\, \Delta x$$

Which is for isotropic conditions becomes

$$-K\left(\frac{\partial h}{\partial y}\Delta x - \frac{\partial h}{\partial y}\Delta y\right)\Delta t = Sv\, \Delta h\, \Delta x$$

and on dividing through with $\Delta x.\Delta t$ and going to the limit

$$\frac{K}{S_v}\left[\left(\frac{\partial h}{\partial y}\right)^2 - \frac{\partial h}{\partial y}\right] = \frac{\partial h}{\partial t}$$

at y = h. Neglecting the square of the piezometric gradient leads to the linearized boundary condition:

$$\frac{\partial h}{\partial t} + \frac{K}{S_v}\frac{\partial h}{\partial y} = 0$$

at y = h.

(iii) At the discharge boundary:

$$0 \leqq y \leqq h_w; \qquad h = h_w; \qquad x = 0$$

$$h_w \leqq y \leqq h_0; \qquad y = h; \qquad x = 0$$

where, the first is control by the water level h_w and the second by atmospheric pressure on the surface of seepage, which is ignored by the Dupuit approximation (Nguyen and Raudkivi, 1983).

Boulton (1954 and 1965; Carslaw and Jaeger, 1959 and Jacob Bear, 1979) used the above approximate boundary conditions to solve the transient flow problem in cylindrical co-ordinates but instead of satisfying boundary conditions (ii) at y =h , he complied with it at $y=h_0$, the initial water level. He stated that the error involved made the calculated drawdown too large but that the error would tend to be cancelled due to another assumption in the solution.

Szabo and McGaig (1968) and Streltsova (1975) used the above equations to solve the anisotropic flow case, using a finite difference model, and found the computed drawdown to be in good agreement with that from an analogue solution. Streltsova (1975) and Szabo and McGaig (1968) introduced a vertical diffusivity term and solved a two-dimensional transient flow problem by Laplace transform. She found that the solutions converged to those from the Dupuit approximation for large values of time but did not elaborate on the estimation of the vertical diffusion length.

Solution for instantaneous drawdown at discharge face:

The solution of

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0 \qquad ...(46)$$

with the boundary conditions (i), (ii), (iii) for $S \cong S_v$ and $t < 0$; $h = h_0$

$$t=0 \rightarrow \begin{cases} x=0; & h=h_w \\ x>0; & h=h_0 \end{cases}$$

Since the drawdown is symmetrical with respect to the centre line of the excavation the general solution would be an even function of x, of the form (Boulton, 1954).

$$h=\int_0^\infty T(a,t)\cosh(ay)\sin(a|x|)\,da \qquad ....(47)$$

where, dummy variable a is used which vanishes on integration and T (a,t) is the transient component, a function of a and time t. It will satisfy the Laplace equation, Eq. (46), boundary condition (i), the symmetry requirement and it must have a finite value at x = ∝. For x > 0; | x | = x; the boundary condition (ii), modified for short time, *i.e.* $y=h_0$ like Boulton's approximation, can be written as:

$$\int_0^\infty\left[\frac{\partial T}{\partial t}\cosh(ah_0)+T(a,t)\frac{K}{S_v}a\sinh(ah_0)\right]\sinh(ax)\,da=0$$

i.e. $\frac{\partial T}{\partial t}\cosh(ah_0)+T(a,t)\frac{K}{S_v}a\sinh(ah_0)=0$

where,

$$T(a,t)=T(a,0)\exp[-(K/S_v)\,a\tanh(ah_0)\,t] \qquad ...(48)$$

T (a,0) can be evaluated from the initial boundary condition t=0, giving the initial water surface level (Boulton, 1954).

$$H(a)=\int_0^\infty T(a,0)\cosh(ah_0)\sinh(ax)\,da$$

which is the Fourier sine transform of [T(a,0) cosh (ah_0)] of which the inverse transform gives:

$$T(a,0)=\frac{2}{\pi}\int_0^\infty\frac{H(x)}{\cosh(ah_0)}\sin(ax)\,dx \qquad ....(49)$$

If for convenience the water level at the excavated pit, after the instantaneous drawdown (t = 0) is taken as the datum, then the initial water surface level H (x) is a Heaviside unit function (Bath, 1968) with a jump of (h_0-h_w) at x = 0 . The integral of Eq. (49) can, therefore, be evaluated as (Boulton, 1954).

$$T(a,0)=\left\{\frac{2}{[\pi\cosh(ah_0)]}\right\}\left[\frac{h_0-h_w}{a}\right] \qquad ....(50)$$

and the solution for the piezometric head with datum at $y = h_w$ becomes:

$$h=\frac{2}{\pi}(h_0-h_w)\int_0^\infty\frac{\cosh(ay)}{\cosh(ah_0)}\frac{\sin(ax)}{a}\exp\left[-\left(\frac{K}{S_v}\right)a\tanh(ah_0)t\right]da \qquad ..(51)$$

When, $y= h_0$ is introduced above, which tends to reduce the error due to boundary condition (ii), Eq. (48), the free surface equation becomes (Boulton, 1954).

$$h=\left(\frac{2}{\pi}\right)(h_0-h_w)\int_0^\infty\left(\frac{1}{a}\right)\sin(ax)\exp\left[-\left(\frac{K}{S_v}\right)a\tanh(ah_0)t\right]da+h_w \qquad ..(52)$$

The drawdown at distance *x* from the pit and time t is :

$$s=h_0-h=(h_0-h_w)\left\{1-\left(\frac{2}{\pi}\right)\int_0^\infty\left(\frac{1}{a}\right)\sin(ax)\exp\left[-\left(\frac{K}{S_v}\right)a\tanh(ah_0)t\right]da\right\} \qquad ....(53)$$

Substitution of $\tau=(\frac{K}{S_v h_0})t$ and $\lambda = ah_0$ in Eq. (53) yields (Boulton, 1954).

$$s=(h_0-h_w)\left\{1-\left(\frac{2}{\pi}\right)\int_0^\infty\left(\frac{1}{\lambda}\right)\sin\left(\frac{\lambda x}{h_0}\right)\exp[-\tau\lambda\tanh\lambda]\,d\lambda\right\} \qquad ..(54)$$

or

$$s=(h_0-h_w)\left[1-G\left(\frac{x}{h_0},\tau\right)\right] \qquad ...(55)$$

where,

$$G\left(\frac{x}{h_0},\tau\right)=\left(\frac{2}{\pi}\right)\int_0^\infty\left(\frac{1}{\lambda}\right)\sin\left(\frac{\lambda x}{h_0}\right)\exp\left[-\tau\lambda\tanh\lambda\right]d\lambda$$

and λ vanishes on integration. Values of the integral $G\left(\frac{x}{h_0},\tau\right)$ have been evaluated numerically.

Comparison with the dupuit approximation :

The dupuit-forchheimer model assumes that the velocity through the saturated depth of the aquifer is constant and proportional to the gradient of the water table at that point, *i.e.* small surface and negligible vertical piezometric gradients as is the case when the drawdown is small. The transient flow Eq. is (Boulton, 1965).

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)=\left(\frac{S_v}{K}\right)\frac{\partial h}{\partial t} \qquad \text{....(56)}$$

Polubarinova-Kochina (1962) solved this equation by two types of linearization and by using the Boltzmann transformation. The linearization $h^2 = h\,h_0$ yielded.

$$h = h_w + (h_0 - h_w)\,\mathrm{erf}\left\{\frac{x}{2}\left[\left(\frac{kh_0}{S_v}\right)t\right]^{\frac{1}{2}}\right\} \qquad \text{....(57)}$$

and the linearization $u = h^2$ led to:

$$h=\left\{h_w^2+\left(h_0^2-h_w^2\right)\mathrm{erf}\left[\frac{x}{2}\right]\left[\left(\frac{kh_0t}{S_v}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \qquad \text{...(58)}$$

These are to be compared to Eq. (55), *i.e.*

$$h = h_w + (h_0 - h_w)\,G\left(\frac{x}{h_0},\tau\right) \qquad \text{....(59)}$$

The G-function plotted against λ, with τ as a parameter, shows that the integral G varies significantly only over small values of λ. For example for $\frac{x}{h_0}=100$ and all values of 0< a:YMKN; $\frac{x}{h_0}=10.0<\lambda<0.33$ both followed by small positive and negative oscillations with diminishing amplitude as λ increases. When $\frac{x}{h_0}=1.0$ and 2: $\geqq$ 2 the range is 0 <: a :< ~ 2; 2: $\geqq$ 4 the range is 0 <: a :< 1. 5 and 2: $\geqq$ 6 the range is 0 < a :< 1.0. For $\frac{x}{h_0}=0.1$, 2: $\geqq$ 4 the range is 0 < a :< 1.5 . In the range of a < 1, the value of $\tanh\lambda\cong\lambda$, particularly when a :<< 1, then (Boulton,1954).

$$G\cong\frac{2}{\pi}\left\{\int_0^\infty\left[\sin\left(\frac{{}^{a}x}{h_0}\right)/{}^{a}\right]\exp\left(-\lambda^2\tau\right)d\lambda\right\} \qquad (60)$$

where, the integral:

$$\int_0^\infty\left[\sin\left(\frac{{}^{a}x}{h_0}\right)/{}^{a}\right]\exp\left(-\lambda^2\tau\right)d\lambda=\left(\frac{\pi}{2}\right)\mathrm{erf}\left[\frac{x/h_0}{2\sqrt{\tau}}\right]$$

as can be shown with the help of tables (Handbook of Mathematical Functions, National Bureau of Standards p. 302, $g(x)=\int_0^\infty\exp\left(-at_x^2\right)\cos 2\,x\,t\,dt=\frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\exp\left(-\frac{x^2}{\alpha}\right)$ and $f(x)=\int_0^\infty g(x)dx$). This substituted in Eq. (59) yields $h\cong h_w+(h_0-h_w)\,\mathrm{erf}\left[\frac{(x/h_0)}{2\sqrt{\tau}}\right]$ which is identical with Eq. (57). The difference between drawdown using G as defined by Eq. (55) and (60) is $\Delta=\frac{2(h_0-h_w)(G_1-G)}{\pi}$

where, G, is as defined by Eq (60).Computer printout shows that G and G_1 are identical to three decimal places and two decimal places when.

x / h_0	3 decimals $\tau >$		2 decimals $\tau >$		
0.01	4		0.5		
0.1	10		4		
1.0	60		8		
3.0	80		20		
10			80		
Both functions have the value 1.000 when					
x / h_0	5	6	7	8	9	10
$\tau \leqq$	0.1	0.5	1.0	1.0	2.0	2.0

It is seen that for small values of τ ($\tau < 0.1$) the Dupuit solutions overestimate the drawdown but in the vicinity of $\tau \geqq 0.1$ the second linearization, Eq. (58), comes close to the above presented solution which progressively moves closer to the solution by the first linearization, Eq. (57) and these are almost identical for $\tau > 2.0$ (Boulton, 1954).

It should be noted that the capillary fringe and the flow through this fringe were ignored both by the above solution and those by Polubarinova-Kochina (1962). The justification here is that large excavations are implied where the influence of the fringe frequently becomes small. The numerical analysis of the transient flow situation, including the capillary fringe and K as a function of elevation, by Vauclin *et al.* (1975) and their laboratory experiments with a drawdown s = 0.7 m, porosity n = 0.3 and saturated soil K = 0.4 m h^{-1} yielded continuously downward sloping water table profiles. The above solution straddles these profiles as an "s-curve", *i.e.* the water table is presented as a wave front which translates away from the cut and flattens with time. For large time increment an outflow region, with a uniform depth of flow, is established.

The flow rate into the excavation per unit length is (Boulton,1954).

$$\mathbf{q} = -2\mathbf{T}\frac{\partial \mathbf{s}}{\partial \mathbf{x}}\bigg|_{x=0} = \frac{2\mathbf{T}(\mathbf{h}_0 - \mathbf{h}_w)}{}\frac{\partial}{\partial \mathbf{x}}\mathbf{G}\left(\frac{\mathbf{x}}{\mathbf{h}_0}, \tau\right)\bigg|_{x=0}$$

$$= \left(\frac{4\mathbf{T}}{\pi \mathbf{h}_0}\right)(\mathbf{h}_0 - \mathbf{h}_w)\int_0^\infty \exp(-\tau\lambda \tan \mathbf{h}\, \lambda)\, d\lambda$$

which for a given time can be evaluated with the aid of the numerically evaluated integral.

Conclusion :

Analytical solutions are given for two-dimensional Laplace equation under various conditions and also transient two-dimensional unconfined groundwater, flow into a large excavation after a rapid lowering of the outflow level (Boulton, 1954). The solution approximates that based on the Dupuit approximation for values of

$$\tau = \left(\frac{\mathbf{Kt}}{\mathbf{S}_v \mathbf{h}_0}\right) > 2 \,.$$

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