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Research Paper :

Rise and decline of water table in response to linearly decreasing replenishment rate

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ABSTRACT

The transient drainage for two layered soil, when the initial water level in the ditches was at interface of the layers has been studied. The flow system in the layered soils has been described by Boussinesq's equation in the form of the Girinsky potential. The analytical solutions were derived to describe rise and decline of water table in response to time varying replenishment rate for linearly decreasing replenishment rate. The Laplace transformation was used to obtain the solution with initial and boundary conditions. The two layered drainage problem was also investigated on a simulated vertical Hele-Shaw model for the validation of theoretical solutions. The comparison of results of observed and computed water table at mid point was found to be in good agreement for the entire duration of the experiments for linearly decreasing replenishment rate. From the comparison, it was also revealed that the solution for layered soil gave more accurate results as compared to homogeneous soil with weighted average hydraulic conductivity (WAHC). Hence, the proposed solutions can be used for the design of drainage system or flow through the aquifer of two layered soil.

Key words : Subsurface drainage, Two layered soil, Drainage, Replenishment rate, Water table

Most of the drainage theories available in literature attempt to describe water table behaviour in response to uniform percolation related to flat lands have been developed by obtaining the solution of partial differential equations derived by Boussinesq's equation (1877, 1904), which is based on Dupuit-Forchheimer assumptions and potential theory. Massland (1959) analysed the problem of water table fluctuation in response to constant recharge, intermittent constant recharge and intermittent instantaneous recharge. Sewa Ram and Chauhan (1987 a and b) obtained transient solutions for water table rise in a sloping aquifer receiving time varying recharge and lying between two parallel ditches reaching up to impermeable layer. The solution was obtained for linearly decreasing replenishment rate with time.

Many investigators such as Chieng and Uziak (1991), Sharma *et al.* (1991), Kumar and Chauhan (1999), Sharma *et al.* (2000) presented analytical solutions for steady or unsteady state condition of layered soil. Some earlier cited workers also conducted experiments on Hele-Shaw model to verify their theoretical investigations. Khan *et al.*, 1989 and Kumar, 1998.

Thus, the objective of the present paper is to develop analytical solution for unsteady state rise and decline of water table under linearly decreasing replenishment rate for two layered soil, when the initial water level in the interface of layers.

Problem formulation:

The assumptions considered for formulating the mathematical problem are given below:

– The soil consists of two layers that are unconfined, homogeneous and isotropic within themselves.

– The phreatic surface lies over a flat impermeable bed.

– Dupuit-Forchheimer assumptions are valid.

– The generalized Boussinesq equation is valid for a stratified aquifer.

– The drainage system consists of equally spaced open ditch drain reaching upto impervious layer.

–The initial water table is at h_0 for t = 0 and $0 \leq x \leq L$.

The flow system in the drains under unsteady state condition is taking place due to variable replenishment rate. The Boussinesq's equation for unsteady state flow, in the form of Girinsky potential is written as below:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{k'}\frac{\partial \theta}{\partial t} - R \qquad (1)$$

where,

R = Recharge rate per unit surface area

$$k' = \frac{1}{f}\int_0^h K(Z) \cdot dz \qquad (2)$$

The Girinsky potential 'θ' is an integral of all the values of potential at all possible positions of the free surface on the vertical line over 0<z<h, as follows:

$$\theta = \int_{o}^{h} (z - h)\ K\ (z)\ \cdot dz \qquad (3)$$

where,

K (z) = hydraulic conductivity of an individual layer

h = height of phreatic surface above the impervious layer.

Girinsky potential for the upper stratum of the problem under study may be:

$$\theta = K_2 \int_{o}^{ho} (z - h) \cdot dz + K_1 \int_{ho}^{h} (h - z) \cdot dz \qquad (4)$$

where,

K_1 and K_2 = hydraulic conductivity of the upper and lower layer of the soil

ho = thickness of the lower layer.

Integrating equation (2) within the region under study is written as:

$$k' = \frac{K_2 h_o + K_1 (h_{av} - h_o)}{f_1} \qquad (5)$$

$$h_{av} = h_o + \frac{h_m}{2}$$

where,

f_1 = drainable porosity of the upper layer

h_m = height of water table mid way between the drains above the water levels in the drains

Boundary conditions:

Initial condition

θ (*x*, o) = q_o at all *x* $0 \le x \le L$

at t = 0

Boundary conditions (6)

θ (o, t) = θ_o at *x* = 0 t > 0

θ (L, t) = θ_o at *x* = L t > 0

The value of θ and θ_o

$$\theta = \frac{-K_1}{2}\left[(h + (C-1)h_o)^2\right] - C(C-1)h_o^2\}$$

$$\theta = \frac{-K_1}{2}\left[(h + B)^2 - d\right] \qquad (7)$$

$$\theta_0 = \frac{-K_1}{2}\left[(h_o)^2\right] \qquad (8)$$

where,

$$d = C(C-1)h_o^2 \qquad B = (C-1)h_o \qquad C = \frac{K_2}{K_1}$$

Analytical solutions:

The partial differential equation in the form of Girinsky potential for linearly decreasing replenishment rate, R (t) = w - mt is written as below:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{k'}\frac{\partial \theta}{\partial t} - (w - mt) \qquad (9)$$

The general solution of equation (11) is written as

$$\bar{\theta}(x,s) = C_1 \cos h\ \sqrt{s/k'}\ x + C_2 \sin h\ \sqrt{s/k'}\ x + \frac{\theta_o}{s} + \frac{wk'}{s^2} - \frac{mk'}{s^3} \qquad .....(10)$$

where, C_1 and C_2 are constant of integration

The constant C_1 and C_2 were calculated using transformation of boundary conditions using equation (6) yields

$$\bar{\theta}\ (o,s) = \theta_o/s$$
$$\bar{\theta}\ (L,s) = \theta_o/s \qquad (11)$$

By putting these transformation in equation (10)

$$C_1 = \frac{mk'}{s^3} - \frac{wk'}{s^2}$$

$$C_2 = \frac{\left(\frac{mk'}{s^3} - \frac{wk'}{s^2}\right)\left(1 + \cos h \sqrt{s/k'}\ L\right)}{\sin h\ \sqrt{s/k'}\ L}$$

Substitute value of C_1 and C_2 in equation (10) and rearranging the term

$$\bar{\theta}(x,s) = \frac{\theta_o}{s} - \frac{Rk'}{s(s+r)}\left[\frac{\sin h \sqrt{s/k'}\ (L-x)}{\sin h \cdot \sqrt{s/k'} \cdot L} + \frac{\sin h\ \sqrt{s/k'}\ x}{\sin h\ \sqrt{s/k'}\ L} - 1\right] \qquad .....(12)$$

By taking inverse transformation the solution of equation (12) may be written as follows:

$$\theta(x,t) = \theta_o - \frac{4L^2}{\pi^3}\left[\left(w + \left(mt - \frac{mL^2}{n^2\pi^2}\right)\right)\right] \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi\pi}{L}\left(1 - e^{-n^2\pi^2 k't/L^2}\right) \quad (13)$$

By putting value of the Girinsky potential, θ and $\theta_{o,}$ in above equation, the general solution of phreatic surface for two-layered soil is expressed by,

$$h = -(C-1)h_o + \sqrt{C(C-1)h_o^2 + Ch_o^2 + \frac{8L^2}{K_1\pi^3}\left[w - \left(mt - \frac{mL^2}{n^2\pi^2}\right)\right]} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi\pi}{L}\left(1 - e^{-n^2\pi^2 k't/L^2}\right) \quad .....(14)$$

The value of h (L/2, t) at mid point between the parallel drain is obtained as

$$h(L/2,\ t) = -(C-1)\, h_o + \sqrt{C(C-1)h_o^2 + Ch_o^2 + \frac{8L^2}{K_1\pi^3}} \left[w - \left(mt - \frac{mL^2}{n^2\pi^2}\right)\right] \sum_{n=1,-3,5}^{\infty} \frac{1}{n^3}\left(1 - e^{-n^2\pi^2 k't/L^2}\right) \quad (15)$$

For the case, when C = 1 *i.e.* homogeneous and isotropic aquifer equation (14) reduce to

$$h^2 = h_o^2 + \frac{8L^2}{K_1\pi^3}\left[w - \left(mt - \frac{mL^2}{n^2\pi^2}\right)\right] \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi\pi}{L}\cdot\left(1 - e^{-n^2\pi^2 k't/L^2}\right) \quad (16)$$

And equation (15) reduce to

$$h^2(L/2,t) = h_o^2 + \frac{8L^2}{K_1\pi^3}\left[w - \left(mt - \frac{mL^2}{n^2\pi^2}\right)\right] \sum_{n=1,-3,5}^{\infty} \frac{1}{n^3}\left(1 - e^{-n^2\pi^2 k't/L^2}\right) \quad (17)$$

METHODOLOGY

The experimental verifications of theoretical solution were conducted on a vertical Hele-Shaw model consisting of two closely parallel plates. It was designed and fabricated in the workshop of College of Technology and assembled in 'Flow Through Porous Media Laboratory' of the Department of Irrigation and Drainage Engineering, Pantnagar. The model was assembled for studies on simulated two layered soils for required boundary conditions.

The recharge nozzles were calibrated by maintaining constant head in the oil channel just before the experiment run. The average outflow per nozzle at 28^0C was found to be same for all 18 nozzles. The linearly decreasing replenishment rate was obtained by decreasing the number of recharge nozzles from 18 at t = 10 min, 16 at t = 20 min and so on up to 2 at t = 90 minute. The replenishment rate was decreased as a step function which resulted in a straight line relationship of the type R (t) = w - mt by joining the mid point of the bars of replenishment rates at subsequent time intervals as shown in Fig. 1.

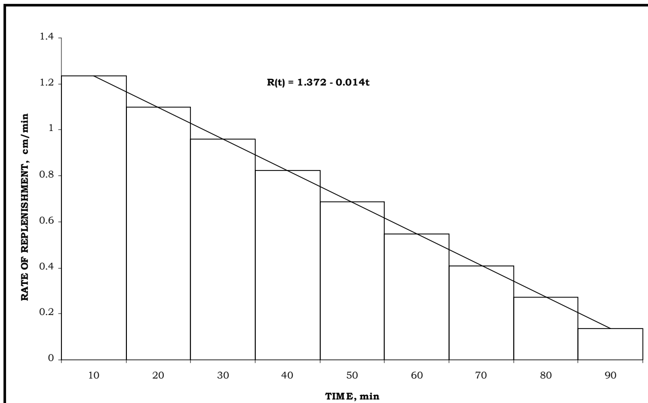

Fig. 1 : Lineraly decreasing replenishment rate

RESULTS AND DISCUSSION

The hydraulic conductivity of model of upper layer was estimated to be 35.32 cm/min and for lower layer as 10.36 cm/min, which provided K_2/K_1 ratio as 0.29. The ratio of replenishment rate to hydraulic conductivity (R/K_1) was estimated to be 0.0349.

Unsteady state rise of water table in response to linearly decreasing replenishment rate:

The variation between observed and computed water table profiles is shown in Fig. 2. The water table levels at mid point for layered soil and with weighted average hydraulic conductivity are shown in Fig. 3 and 4, respectively.

From Fig. 2 to 4, it was observed that the average deviation near the ditch was 4.3% and at the mid point 0.79%. The deviation near the ditch for rising table was

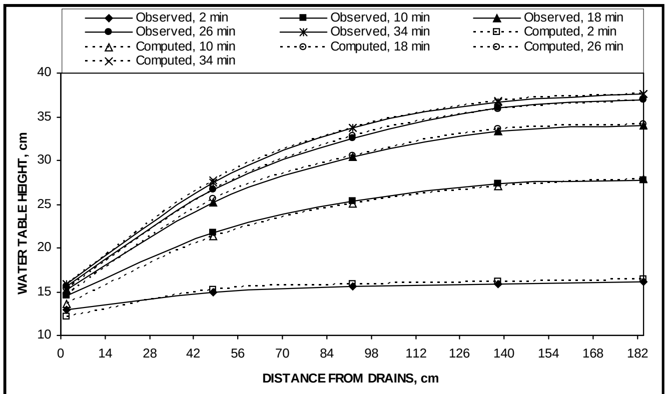

Fig. 2 : Observed and computed rising water table profiles for exponentially declining replenishment rate

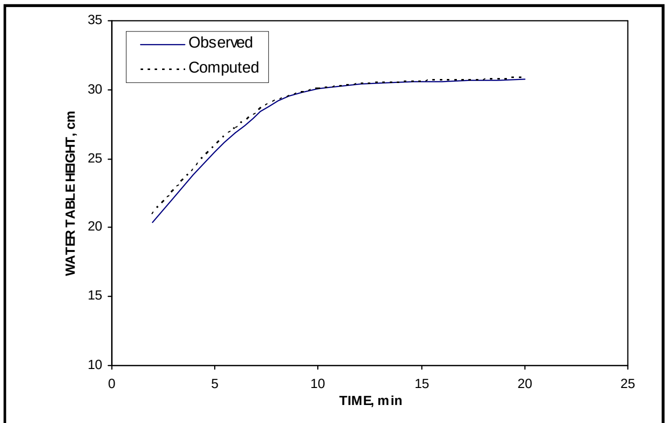

Fig. 3 : Rising water table at mid point for layered soil with exponentially declining eplenishment rate

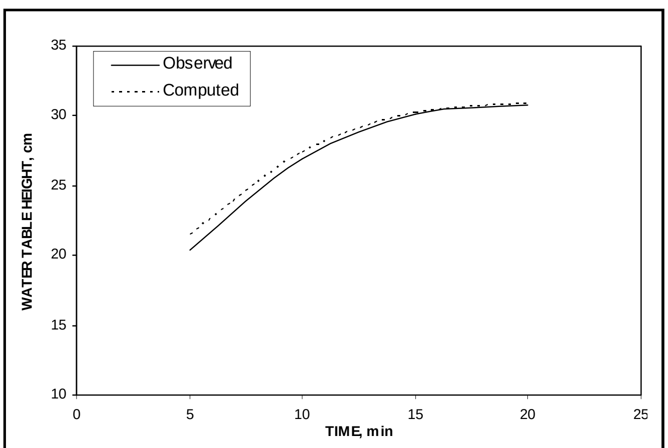

Fig. 4 : Rising water table at mid point for weighted average hydraulic conductivity with linearly declining replenishment rate

more than the deviation at mid point and decreased with increase in time. The deviations at mid point near the ditch for layered soil ranged from 0.16 to 1.43% and with weighted average hydraulic conductivity from 1.18 to 2.19%. It showed that the developed solution for two layered soil with linearly decreasing replenishment rate gave more accurate result than the corresponding homogeneous solution with weighted average hydraulic conductivity.

Unsteady state decline of water table in response to linearly decreasing replenishment rate:

The observed and computed falling water table profiles are shown in Fig. 5. The water table at mid point for layered soil and corresponding values with weighted average hydraulic conductivity are shown in Fig. 6 and 7, respectively.

It was observed that the average deviation near the ditch was 3.79% and at mid point 0.51%. The deviation near the ditch was more and increased with time. The deviation at mid point for layered soil ranged from 0.0 to 1.22% and with weighted average hydraulic conductivity from 0.97 to 7.56%. At the mid point, declining of water table was having lesser deviation as compared to near

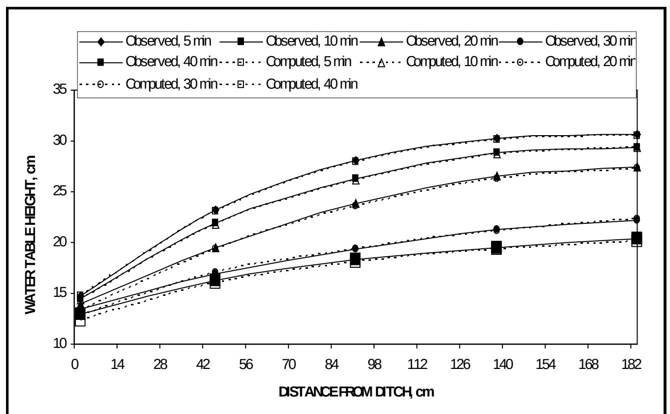

Fig. 5 : Observed and computed falling water table profiles for linearly declining eplenishment rate

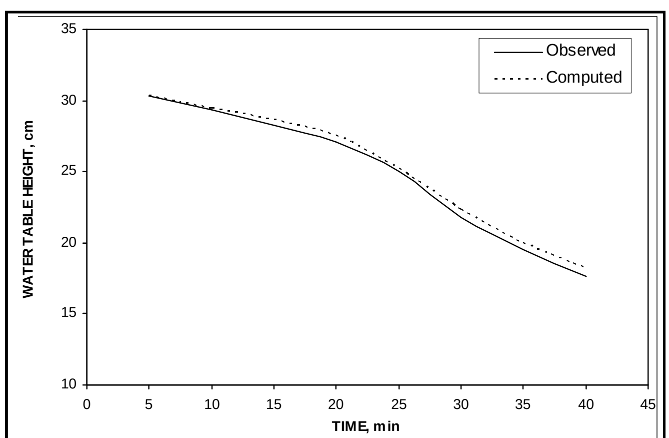

Fig. 6 : Falling water table at mid point for layered soil with linearly declining replenishment rate

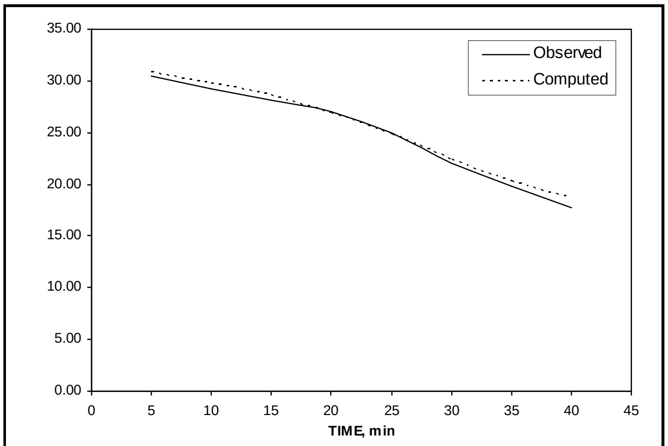

Fig. 7 : Falling water table at point for weighted average hydraulic conductivity with linearly declining replenishment rate

the ditch and layered soil solution gave more accurate results. In the present study, a similar trend of deviation was observed and errors were found to be well within the acceptable range. Thus, the proposed analytical solutions predicted reasonably correct water table close to the experimental results.

Conclusion:

The problem of transient drainage for two layered soil, when the initial water level in the ditches is at interface of the layers, has been studied. The flow system in the layered soils has been described by Boussinesq's equation in the form of the Girinsky potential. The analytical solutions for unsteady state rise and decline of water table for two layered soil, for linearly decreasing replenishment rate compared with laboratory experiments on simulated viscous fluid vertical Hele-Shaw model. The model fluid was used HP 140 oil with kinematic viscosity of 8 stokes at 28^0C. The proposed mathematical models for layered soils are considerably accurate for practical use in the design of subsurface drainage systems. The following conclusions were drawn in this study.

The deviation of rising water table for linearly decreasing replenishment rate was less at mid point as compared to near the ditch and decreased with increase in time. The deviation of falling water table for linearly decreasing replenishment rate was less at mid point as compared to near the ditch and increased with increase in time. The deviation for linearly decreasing replenishment rate of solution for homogeneous soil with weighted average hydraulic conductivity at mid point was found to be more in comparison to the solution for layered soil. It reveals that, the solution for layered soil gives relatively accurate results as compared to the solution for homogeneous soil with weighted average hydraulic conductivity. The error at mid point between ditches for rising and falling water table for linearly decreasing replenishment rate were found to be within the reasonable limits for a practical use of the proposed analytical solutions under an unsteady state condition.

Notations:

θ = Girinsky potential function

θ (0, t) = Girinsky potential at x = 0

θ (L, t) = Girinsky potential at x = L

b = spacing between two plates, L

f = Drainable porosity, dimensionless

L = Spacing between drains, L

r = Coefficient in the time varying exponentially declining replenishment rate, T^-

R (t) = Time varying replenishment rate per unit surface area of the aquifer, LT^{-1}

s = parameter of the Laplace transformation

t = Time, T

x = horizontal distance from a reference point L

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